

EE 477  
Digital Signal Processing

7a

z-Transforms

# z-Transform Definition

- For a finite length signal  $x[n]$ ,  $n=0\dots N$ , the z-transform is defined:

$$\begin{aligned} X(z) &= \sum_{k=0}^N x[k] z^{-k} \\ &= x[0]z^0 + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \dots + x[N]z^{-N} \end{aligned}$$

- $z$  is simply a variable representing any *complex number*, that is,  $z$  is a *complex variable*
- $X(z)$  is customarily expressed as a ratio of polynomials in  $z^{-k}$

# $n$ -domain and $z$ -domain

- Given  $x[n]$ , we can use the formula to “take the  $z$ -transform” and get  $X(z)$ .
- Given  $X(z)$ , we can “take the inverse  $z$ -transform” and get  $x[n]$ .
- Examples:

$$x[n] = \delta[n - n_0] \Leftrightarrow X(z) = z^{-n_0}$$

$$x[n] = 3\delta[n] - 6\delta[n - 1] + 2\delta[n - 2] + 7\delta[n - 3]$$

$$\Leftrightarrow X(z) = 3 - 6z^{-1} + 2z^{-2} + 7z^{-3}$$

# z-transforms and Linear Systems

- Consider FIR filter (convolve h and x):

$$y[n] = \sum_{k=0}^M h[k] x[n-k] = \sum_{k=0}^M b_k x[n-k]$$

- We previously applied  $x[n]=e^{j\omega n}$ , all n, to define the *frequency response*.
- Now apply  $x[n]=z^n$  :

$$y[n] = \sum_{k=0}^M b_k z^{(n-k)} = \sum_{k=0}^M b_k z^n z^{-k} = \left( \sum_{k=0}^M b_k z^{-k} \right) \underbrace{\left\{ z^n \right\}}_{x[n]}$$

$H(z)$

# System Function

- Note the last result:  $H(z) = \sum_{k=0}^M b_k z^{-k} = \sum_{k=0}^M h[k] z^{-k}$   
The system function  $H(z)$  is the  $z$ -transform of the unit sample response.
- System function (product) is related to convolution, since if  $x$  is  $z^n$

$$\begin{aligned} y[n] &= H(z) z^n \\ &= h[n] * z^n \end{aligned}$$

# z-transform properties

- Linearity:  
 $Z\{ax_1[n]+bx_2[n]\}=aZ\{x_1[n]\}+bZ\{x_2[n]\}$
- Delay: multiply by  $z^{-1}$  corresponds to delay of one sample; multiply by  $z^{-n_0}$  corresponds to delay of  $n_0$ .
- General infinite length z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

*Note: need to verify that the sum converges!!*

# Z-transform and convolution

- Take the z-transform of a convolution expression:

$$y[n] = x[n] * h[n] = \sum_{k=0}^M h[k]x[n-k]$$

$$Y(z) = Z\{h[0]x[n] + h[1]x[n-1] + \dots + h[M]x[n-M]\}$$

$$= \sum_{k=0}^M h[k](z^{-k} X(z)) = \left( \sum_{k=0}^M h[k]z^{-k} \right) X(z)$$

$$= H(z)X(z)$$

*Sequence domain convolution  $\Leftrightarrow$  z-domain product*

# z-transform: polynomial in $z^{-1}$

- Consider an FIR function:

$$y[n] = x[n] - 8.5x[n-1] + 13.5x[n-2] + 9x[n-3]$$

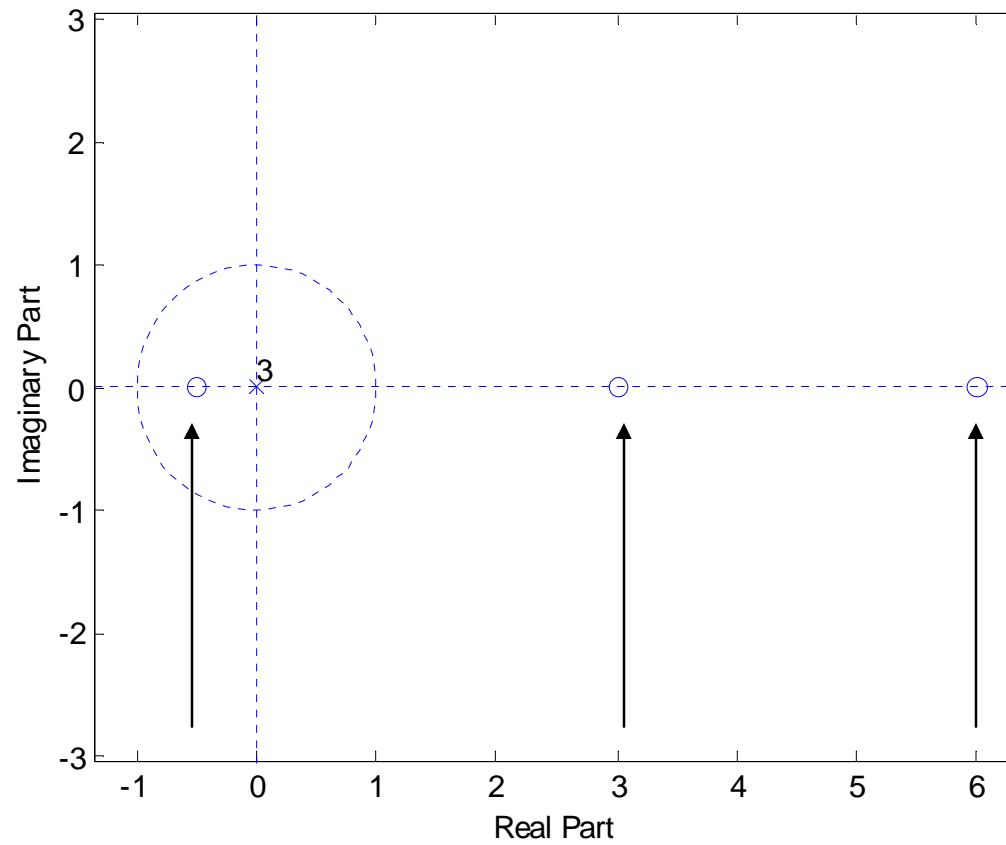
$$Y(z) = X(z) - 8.5z^{-1}X(z) + 13.5z^{-2}X(z) + 9z^{-3}X(z)$$

$$\begin{aligned} H(z) &= Y(z) / X(z) = 1 - 8.5z^{-1} + 13.5z^{-2} + 9z^{-3} \\ &= (1 + 0.5z^{-1})(1 - 3z^{-1})(1 - 6z^{-1}) \end{aligned}$$

- Zeros of  $H(z)$ :  $-.5, 3, 6$ ; 3 poles @ 0



# Graphical Depiction of $H(z)$



# Another $h[n] \leftrightarrow H(z)$ example

- Consider an FIR system:

$$h[n] = \delta[n] - 2\delta[n-1] + 5\delta[n-2] + 5\delta[n-3] - 2\delta[n-4] + \delta[n-5]$$

$$H(z) = 1 - 2z^{-1} + 5z^{-2} + 5z^{-3} - 2z^{-4} + z^{-5}$$

$$= (1 - 2.618z^{-1} + 6.8541z^{-2})(1 - 0.382z^{-1} + 0.1459z^{-2})(1 + z^{-1})$$

- System zeros:

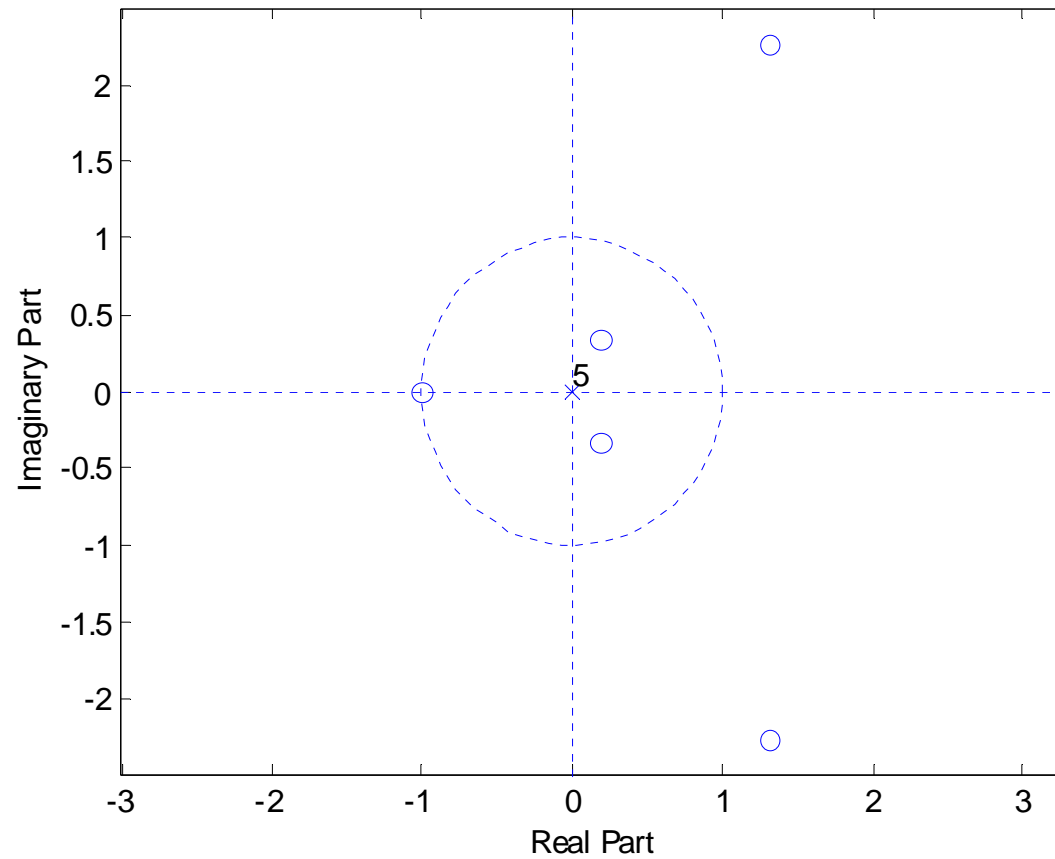
$$1.3090 + j2.2673, \quad 1.3090 - j2.2673$$

$$-1.0000$$

$$0.1910 + j0.3308, \quad 0.1910 - j0.3308$$

# Z-plane Sketch

Complex conjugate zeros  $\Rightarrow$  real filter coefficients



# Example z-transform product

- **Ex:**  $x[n] = \delta[n-1] - 5\delta[n-2] + 8\delta[n-8] - 4\delta[n-9]$   
 $h[n] = \delta[n] + \delta[n-1] - 0.3\delta[n-2]$
- **Compute using z-transform product:**

$$X(z) = z^{-1} - 5z^{-2} + 8z^{-8} - 4z^{-9}$$

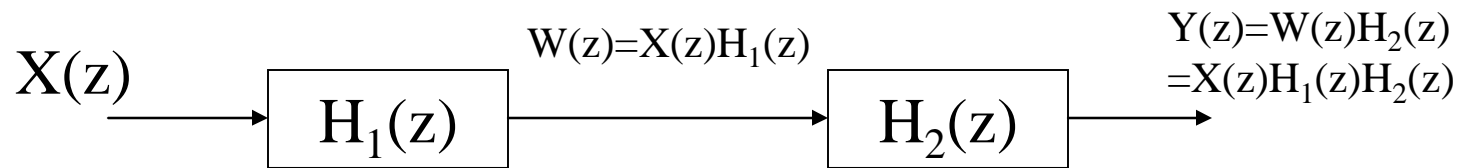
$$H(z) = 1 + z^{-1} - 0.3z^{-2}$$

$$Y(z) = X(z)H(z)$$

$$= z^{-1} - 4z^{-2} - 5.3z^{-3} + 1.5z^{-4} + 8z^{-8} + 4z^{-9} - 6.4z^{-10} + 1.2z^{-11}$$

# Cascade of Systems

- Note cascade in terms of system functions:



- System function of cascade is the *product* of the system functions