EELE 477 Digital Signal Processing

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FIR Frequency Response

FIR response to sinusoids

The general definition of FIR:

$$y[n] = \sum_{k=0}^{M} b_k \cdot x[n-k] = \sum_{k=0}^{M} h[k] \cdot x[n-k]$$

• What if input is complex exponential?

$$x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$$

$$y[n] = \sum_{k=0}^{M} b_k Ae^{j\phi}e^{j\hat{\omega}(n-k)}$$

$$= Ae^{j\phi}e^{j\hat{\omega}n}\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

$$= Ae^{j\phi}e^{j\hat{\omega}n}H(\hat{\omega})$$

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Frequency Response

Note the result carefully:

$$y[n] = Ae^{j\phi}e^{j\hat{\omega}n}H(\hat{\omega})$$

<u>IF</u> the input is a <u>complex exponential</u>, the output is a complex exponential with the same frequency, but in general a different amplitude and phase as determined by *H*(ω): the frequency response.

Frequency response (cont.)

 For FIR systems, the frequency response is determined by the coefficient sequence (which is just the impulse response sequence).

$$H(\hat{\omega}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

• The frequency response is a complex value for a particular frequency.

Frequency response (cont.)

Polar formulation:

$$y[n] = Ae^{j\phi}e^{j\hat{\omega}n}H(\hat{\omega})$$
$$= |H(\hat{\omega})|A \cdot e^{j(\angle H(\hat{\omega}) + \phi)}e^{j\hat{\omega}n}$$

$$|H(\hat{\omega})| = \sqrt{(real)^2 + (imag)^2}$$

$$\angle H(\hat{\omega}) = \arctan\left(\frac{imag}{real}\right)$$

Frequency Response (cont.)

• Example:

$$y[n] = x[n] + 4 \cdot x[n-1] + 3 \cdot x[n-2]$$

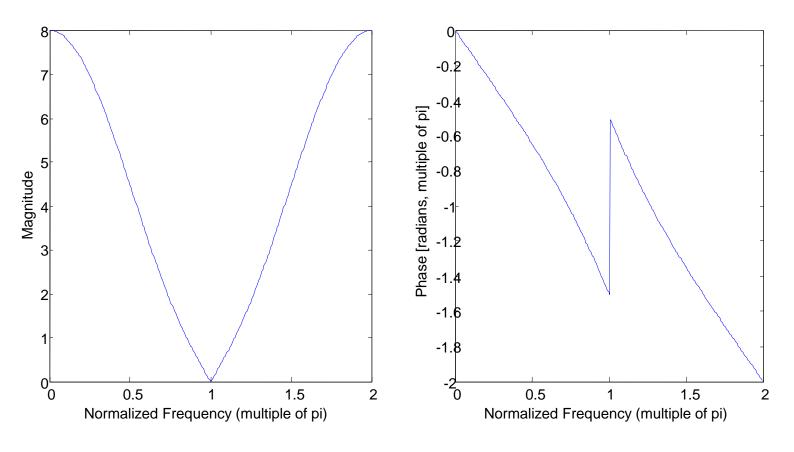
$$\{b_k\} = \{1, 4, 3\}$$

$$H(\hat{\omega}) = 1 + 4e^{-j\hat{\omega}} + 3e^{-j2\hat{\omega}}$$

$$|H(\hat{\omega})| = \left[(1 + 4\cos\hat{\omega} + 3\cos 2\hat{\omega})^2 + (4\sin\hat{\omega} + 3\sin 2\hat{\omega})^2 \right]^{\frac{1}{2}}$$

$$\angle H(\hat{\omega}) = \arctan\left(\frac{-4\sin\hat{\omega} - 3\sin 2\hat{\omega}}{1 + 4\cos\hat{\omega} + 3\cos 2\hat{\omega}}\right)$$

Frequency Response (cont.)



Superposition

- If the input can be expressed as the sum of complex exponential signals, use the frequency response to determine the individual outputs, then add them up.
- This allows response determination in the frequency domain.

Transient and Steady State

- Note that our complex exponential is doubly infinite: all values of n
- Any practical system will need to start and then (probably) stop later
- Consider:

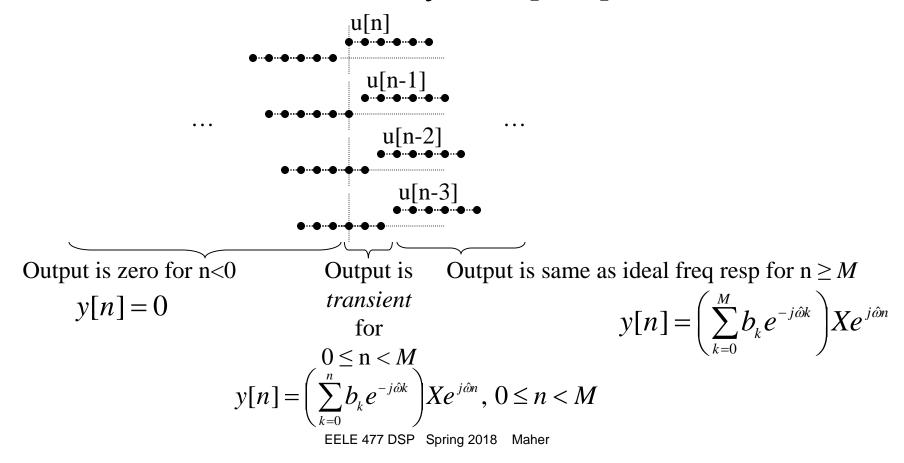
(probably) stop later
$$unit step: u[n] = \begin{cases}
1, & n \ge 0 \\
0, & n < 0
\end{cases}$$

$$x[n] = Xe^{j\hat{\omega}n}u[n]$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} b_k X e^{j\hat{\omega}(n-k)} u[n-k]$$

Transient Response (cont.)

Note the M+1 delayed u[n-k]:



Freq. Response Properties

• For FIR: $h[k] = b_k$, $0 \le k \le M$

$$H(\hat{\omega}) = \sum_{k=0}^{M} h[k] e^{-j\hat{\omega}k}$$

- Note that $H(\omega)$ is always periodic in 2π
- If FIR coefs b_k are real, this implies that $H(\omega)$ has conjugate symmetry:

$$H(-\hat{\omega})=H^*(\hat{\omega})$$

Conjugate Symmetry

- Conjugate symmetry $H(-\hat{\omega}) = H^*(\hat{\omega})$ indicates that the negative frequency portion of the spectrum is the complex conjugate of the positive frequency portion
- If we know one, we can calculate the other

Conjugate Symmetry Proof

$$H^*(\hat{\omega}) = \left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}\right)^*$$

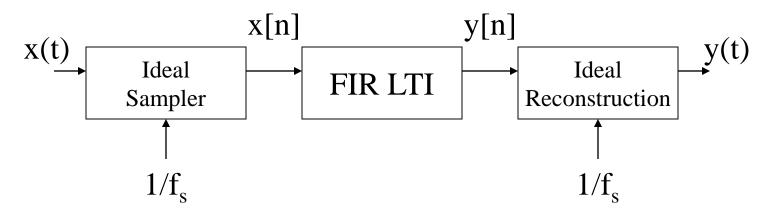
$$= \sum_{k=0}^{M} b_k^* e^{+j\hat{\omega}k}$$
 $= \sum_{k=0}^{M} b_k e^{-j(-\hat{\omega})k} = H(-\hat{\omega})$

Magnitude is an even function, Phase is odd

Real part is even, imaginary part is odd

Discrete time processing of continuous time signals

 Sample a continuous-time signal, perform discrete-time processing, then reconstruct the continuous-time signal



Discrete-time processing (cont.)

Effect of sampling: assume

$$x(t) = Xe^{j\omega t}$$
, sample at $t = nT_s$
 $x[n] = Xe^{j\omega nT_s} = Xe^{j\hat{\omega}n}$, $\hat{\omega} = \omega T_s$
 $y[n] = H(\hat{\omega})Xe^{j\hat{\omega}n} = H(\omega T_s)Xe^{j\omega nT_s}$
 $y(t) = H(\omega T_s)Xe^{j\omega t}$

• Overall response behaves like a continuoustime system with response $H(\omega T_s)$