EELE 477
Digital Signal Processing

7

DTFT

The “frequency response”

- The Discrete-Time Fourier Transform (DTFT)

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

- For a causal system with unit sample response *h*[*n*]:

$$H(e^{j\hat{\omega}}) = \sum_{n=0}^{\infty} h[n]e^{-j\hat{\omega}n}$$

FIR DTFT

- If the system is FIR, $h[n]$ is of finite length N, so:

$$H(e^{j\hat{\omega}}) = \sum_{n=0}^{N-1} h[n] e^{-j\hat{\omega}n}$$

Basic DTFT Properties

- Consider:

$$H(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\hat{\omega}n} = 1$$

- What is DTFT($\delta[n-n_o]$)?

$$H(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} \delta[n-n_0] e^{-j\hat{\omega}n} = e^{-j\hat{\omega}n_0}$$

DTFT Properties (cont.)

- What is DTFT($a^n\, u[n]$)?

$$H(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\hat{\omega}n} = \sum_{n=0}^{\infty} a^n e^{-j\hat{\omega}n} = \sum_{n=0}^{\infty} \left(a e^{-j\hat{\omega}}\right)^n$$

- A sum of an infinite geometric series:

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$$

- (if $|\alpha|<1$)

DTFT Properties (cont.)

- So DTFT($a^n u[n]$), |a|<1:

$$\sum_{n=0}^{\infty}\left(ae^{-j\hat{\omega}}\right)^n = \frac{1}{1-ae^{-j\hat{\omega}}}$$

- Sum of a finite geometric series:

$$\sum_{n=n_0}^{n_1}\alpha^n = \frac{\alpha^{n_0}-\alpha^{n_1+1}}{1-\alpha}$$

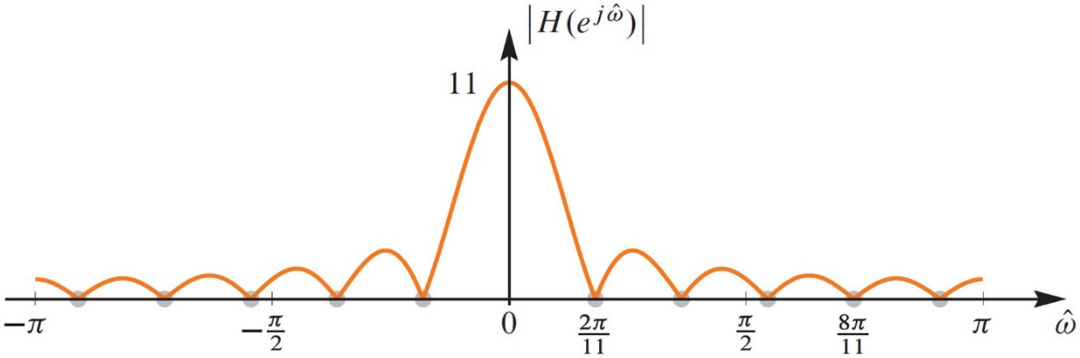
DTFT (cont.)

- So DTFT of a rectangular pulse

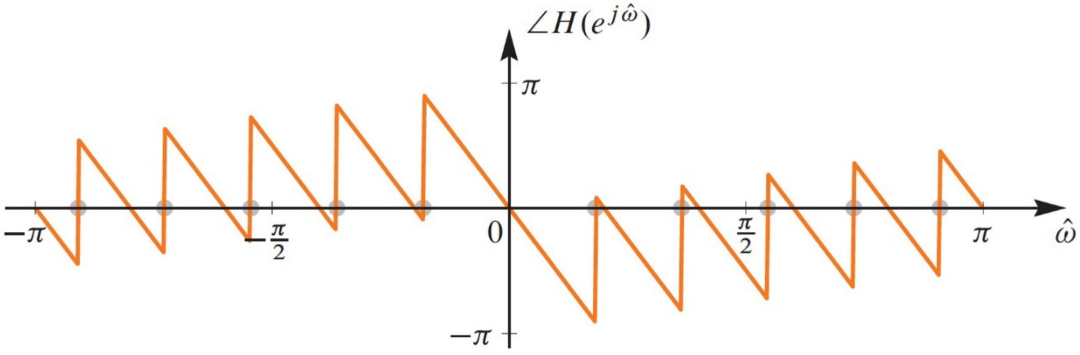
  $r_L[n] = u[n]-u[n-L]$

$$R(e^{j\hat{\omega}}) = \sum_{n=o}^{L-1} 1 e^{-j\hat{\omega}n} = \frac{1-e^{-j\hat{\omega}L}}{1-e^{-j\hat{\omega}}} = \frac{\sin\left(L\hat{\omega}/2\right)}{\sin\left(\hat{\omega}/2\right)} e^{-j\hat{\omega}(L-1)/2}$$

11-point rectangle



(a)



(b)

Table 7-1 Basic discrete-time Fourier transform pairs.

Table of DTFT Pairs	
Time-Domain: $x[n]$	***Frequency-Domain:*** $X(e^{j\hat{\omega}})$
$\delta[n]$	1
$\delta[n-n_d]$	$e^{-j\hat{\omega}n_d}$
$r_L[n] = u[n] - u[n-L]$	$\dfrac{\sin(\frac{1}{2}L\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}\, e^{-j\hat{\omega}(L-1)/2}$
$r_L[n]\, e^{j\hat{\omega}_0 n}$	$\dfrac{\sin(\frac{1}{2}L(\hat{\omega}-\hat{\omega}_o))}{\sin(\frac{1}{2}(\hat{\omega}-\hat{\omega}_o))}\, e^{-j(\hat{\omega}-\hat{\omega}_o)(L-1)/2}$
$\dfrac{\sin(\hat{\omega}_b n)}{\pi n}$	$\begin{cases} 1 & \lvert\hat{\omega}\rvert \le \hat{\omega}_b \\ 0 & \hat{\omega}_b < \lvert\hat{\omega}\rvert \le \pi \end{cases}$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{1 - a e^{-j\hat{\omega}}}$
$-b^n u[-n-1] \quad (\lvert b\rvert > 1)$	$\dfrac{1}{1 - b e^{-j\hat{\omega}}}$

Table 7-2 Basic discrete-time Fourier transform properties.

Table of DTFT Properties		
Property Name	***Time-Domain: $x[n]$***	***Frequency-Domain: $X(e^{j\hat{\omega}})$***
Periodic in $\hat{\omega}$		$X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}})$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(e^{j\hat{\omega}}) + bX_2(e^{j\hat{\omega}})$
Conjugate Symmetry	$x[n]$ is real	$X(e^{-j\hat{\omega}}) = X^*(e^{j\hat{\omega}})$
Conjugation	$x^*[n]$	$X^*(e^{-j\hat{\omega}})$
Time-Reversal	$x[-n]$	$X(e^{-j\hat{\omega}})$
Delay	$x[n-n_d]$	$e^{-j\hat{\omega}n_d}X(e^{j\hat{\omega}})$
Frequency Shift	$x[n]e^{j\hat{\omega}_0 n}$	$X(e^{j(\hat{\omega}-\hat{\omega}_0)})$
Modulation	$x[n]\cos(\hat{\omega}_0 n)$	$\frac{1}{2}X(e^{j(\hat{\omega}-\hat{\omega}_0)}) + \frac{1}{2}X(e^{j(\hat{\omega}+\hat{\omega}_0)})$
Convolution	$x[n] * h[n]$	$X(e^{j\hat{\omega}})H(e^{j\hat{\omega}})$
Autocorrelation	$x[-n] * x[n]$	$\lvert X(e^{j\hat{\omega}})\rvert^2$
Parseval's Theorem	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\hat{\omega}})\rvert^2 d\hat{\omega}$	

Freq. Response Properties

- For FIR: $h[k] = b_k, 0 \le k \le M$

$$H(\hat{\omega}) = \sum_{k=0}^{M} h[k] e^{-j\hat{\omega}k}$$

- Note that *H*(*ω*) is always periodic in 2π
- If FIR coefs b_k are real, this implies that *H*(*ω*) has *conjugate symmetry:*

$$H(-\hat{\omega}) = H^*(\hat{\omega})$$

Conjugate Symmetry Proof

$$H^*\left(e^{j\hat{\omega}}\right) = \left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}\right)^*$$

$$= \sum_{k=0}^{M} b_k^* e^{+j\hat{\omega}k}$$

(b_k are real)

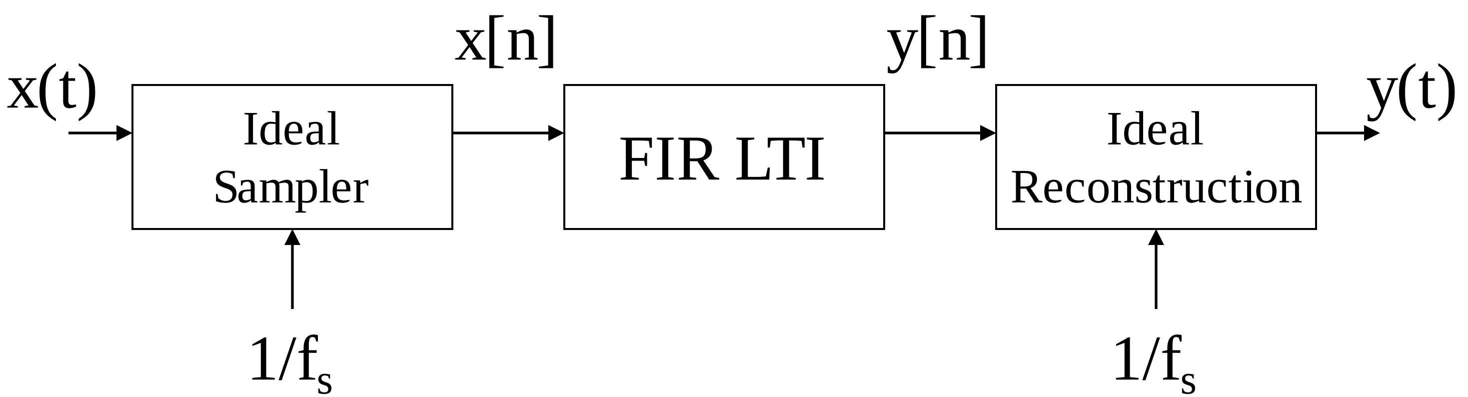
$$= \sum_{k=0}^{M} b_k e^{-j(-\hat{\omega})k} = H\left(e^{-j\hat{\omega}}\right)$$

Magnitude is an even function, Phase is odd

Real part is even, imaginary part is odd

Discrete time processing of continuous time signals

- Sample a continuous-time signal, perform discrete-time processing, then reconstruct the continuous-time signal



Discrete-time processing (cont.)

- Effect of sampling: assume

$$x(t) = Xe^{j\omega t}, \ \text{sample at } t = nT_s$$

$$x[n] = Xe^{j\omega nT_s} = Xe^{j\hat{\omega}n}, \ \hat{\omega} = \omega T_s$$

$$y[n] = H(e^{j\hat{\omega}})Xe^{j\hat{\omega}n} = H(e^{j\omega T_s})Xe^{j\omega nT_s}$$

$$y(t) = H(e^{j\omega T_s})Xe^{j\omega t}$$

- Overall response behaves like a continuous-time system with response $H(\omega T_s)$